Lab 7. Random vectors. Joint distributions.

sgr. 3

Exercise 42. We consider a box containing 5 balls numbered from 1 to 5. We simultaneously take out three balls from the box. We denote X the smallest number obtained. Give the distribution law for the random variable X. Compute the expected value $E(X)$ and the standard deviation $\sigma(X)$. We denote Y the largest number obtained. Are the random variables X and Y independent? Compute the correlation coefficient $\rho(X,Y)$.

$1, 2, 3, 4, 5$ $\qquad \Omega = \{ \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{2,3,4\} \dots \}$

$x = 3 \Rightarrow y = 5 \Rightarrow X, Y$ are not indep.

$P(X=x, Y=y) \neq P(X=x) \cdot P(Y=y) \Longleftrightarrow X, Y$ - are not indep.

$P(X=x / Y=y) \neq P(X=x) \Longleftrightarrow$ — " —

$$X : \begin{pmatrix} 1 & 2 & 3 \\ \frac{6}{\binom{5}{3}} & \frac{3}{\binom{5}{3}} & \frac{1}{\binom{5}{3}} \end{pmatrix} \qquad \binom{5}{3} = \frac{5!}{3! \cdot 2!} = \frac{4 \cdot 5}{2} = 10$$

$$X : \begin{pmatrix} 1 & 2 & 3 \\ \frac{3}{5} & \frac{3}{10} & \frac{1}{10} \end{pmatrix} \qquad E[X] = 1 \cdot \frac{3}{5} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} = \frac{15}{10} = \frac{3}{2}$$

$$V[X] = E[X^2] - (E[X])^2 = \frac{27}{10} - \frac{9}{4} = \frac{9}{20} \Rightarrow \sigma(X) = \sqrt{V(X)} = \frac{3}{2\sqrt{5}}$$

Y - maximum nr. obtained

$$Y: \begin{pmatrix} 3 & 4 & 5 \\ \frac{1}{10} & \frac{3}{10} & \frac{6}{10} \end{pmatrix}$$

$$E[Y] = 3 \cdot \frac{1}{10} + 4 \cdot \frac{3}{10} + 5 \cdot \frac{6}{10} = \frac{45}{10} = 4.5 \left(\frac{9}{2}\right); \quad E[Y^2] = \frac{9}{10} + \frac{16 \cdot 3}{10} + \frac{25 \cdot 6}{10} = \frac{207}{10}$$

$$V[Y] = E[Y^2] - E^2[Y] = \frac{207}{10} - \frac{81}{4} = \frac{414 - 405}{20} = \frac{9}{20} \Rightarrow \sigma(Y) = \frac{3}{2\sqrt{5}}$$

$$X: \begin{pmatrix} 1 & 2 & 3 \\ \frac{3}{5} & \frac{3}{10} & \frac{1}{10} \end{pmatrix}$$

$$X \cdot Y: \begin{pmatrix} 1 \cdot 3 & 1 \cdot 4 & 1 \cdot 5 & 2 \cdot 3 & 2 \cdot 4 & 2 \cdot 5 & 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 5 \\ \frac{1}{10} & \frac{2}{10} & \frac{3}{10} & 0 & \frac{1}{10} & \frac{2}{10} & 0 & 0 & \frac{1}{10} \end{pmatrix}$$

$$P(X=1, Y=3) = \frac{1}{10}$$

$$X \cdot Y: \begin{pmatrix} 3 & 4 & 5 & 8 & 10 & 15 \\ \frac{1}{10} & \frac{2}{10} & \frac{3}{10} & \frac{1}{10} & \frac{2}{10} & \frac{1}{10} \end{pmatrix}, \quad E(X \cdot Y) = \frac{3}{10} + \frac{8}{10} + \frac{15}{10} + \frac{8}{10} + \frac{20}{10} + \frac{15}{10} = \frac{69}{10}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma(X) \cdot \sigma(Y)} = \frac{\frac{3}{20}}{\frac{3}{2\sqrt{5}} \cdot \frac{3}{2\sqrt{5}}} = \frac{3}{20} \cdot \frac{20}{9} = \frac{1}{3}$$

$$Cov(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y) = \frac{69}{10} - \frac{3}{2} \cdot \frac{9}{2} = \frac{3}{20}$$

$$\begin{array}{r} 138 - \\ 135 \\ \hline 3 \end{array}$$

2 4 5

1 2 5
3
4 } 3

Exercise 54. An archer hits a bull's-eye with the probability of 0.09 and the results of different attempts can be taken as independent of each other. If the archer shoots 9 arrows, calculate the probability that:

Exercise 62. Consider the same situation as in Ex. 54. Suppose that the archer misses the target completely with a probability of 0.12. If the archer shoots eight arrows whose performances are independent of each other, what is the probability that (a) the archer scores exactly two bull's-eyes and misses the target exactly once; (b) the archer scores exactly one bull's-eye and misses the target exactly twice. What is the expected number of times the archer misses the target?

X_1 - nr. of bull's-eyes scored

X_2 - nr. of hits inside the target, but are not bull's-eyes

X_3 - nr. of misses

$X = (X_1, X_2, X_3) \sim Mult(n, p)$, $n = 8$

$p = (p_1, p_2, p_3)$ $\quad p_1 = 0.09$ (probab. of a bull's-eye)

$p_2 = 1 - p_1 - p_2 = 1 - 0.09 - 0.12 = 1 - 0.21 = \underline{0.79}$

$p_3 = 0.12$

a) $P(X = (\underset{n_1}{2}, \underset{n_2}{5}, \underset{n_3}{1})) = \dfrac{n!}{n_1! \cdot n_2! \cdot n_3!} \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3} = \dfrac{\overset{3}{\cancel{6}} \cdot 7 \cdot 8}{\cancel{2}! \cdot \cancel{5}! \cdot 1!} \cdot 0.09^2 \cdot 0.79^5 \cdot 0.12^1 = 168 \cdot 0.09^2 \cdot 0.79^5 \cdot 0.12 =$
$= 0.05$

$n \cdot p_3 = E[X_3]$

$X_3 \sim Bin(n, p_3)$

$E[X_3] = 8 \cdot 0.12 = 0.96$

b) $P(X=(1,5,2)) = \frac{8!}{1! \cdot 5! \cdot 2!} \cdot 0.09^1 \cdot 0.79^5 \cdot 0.12^2 = 168 \cdot 0.09 \cdot 0.79^5 \cdot 0.12^2 = \underline{0.067}$

Exercise 80. The lifetime of an electronic tube is a continuous random variable X with exponential density $f(x) = b\,e^{-bx}$ $x \geq 0$. Find $P(k \leq X \leq k+1)$ and the expected lifetime of the tube.

→ homework

Exercise 81. Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is the probability that the repair time exceeds 4 hours? If it exceeds 4 hours what is the probability that it exceeds 8 hours? What is the expected time required to repair the car?

Exercise 85. Calculate the following probabilities both exactly and by using a normal approximation:
(a) $P(X \geq 8)$ where $X \sim Bin(10, 0.7)$
(b) $P(2 \leq X \leq 7)$ where $X \sim Bin(15, 0.3)$
(c) $P(X \leq 4)$ where $X \sim Bin(9, 0.4)$
(d) $P(8 \leq X \leq 11)$ where $X \sim Bin(14, 0.6)$

→ homework

Exercise 86. Suppose that a fair coin is tossed n times. Estimate the probability that the proportion of heads obtained lies between 0.49 and 0.51 for $n = 100$, 200, 500, 1000 and 2000.

a) $X \sim Bin(\underset{n}{\underset{\shortparallel}{10}}, \underset{p}{\underset{\shortparallel}{0.7}}) \Rightarrow X \sim N(\mu, \sigma)$

$\mu = E[X] = np = 10 \cdot 0.7 = 7$

$\sigma = \sqrt{V(X)} = \sqrt{np(1-p)} = \sqrt{10 \cdot 0.7 \cdot 0.3} = 1.45$

$P(X \geq 8) = \int_8^{\infty} f(x)\,dx$, $\quad f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/(2\sigma^2)}$

$= 1 - F(8)$, $\quad F(8) = \int_{-\infty}^{8} f(x)\,dx$

$= 1 - \text{pnorm}(8, \text{mean} = 7, \text{sd} = 1.45) = 0.245$

$P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10) =$

$= \sum_{k=8}^{10} \binom{n}{k} p^k (1-p)^{n-k}$

b) $X \sim Bin(15, 0.3)$, $X \sim N(\mu, \sigma)$, $\mu = 15 \cdot 0.3 = 4.5$, $\sigma = \sqrt{15 \cdot 0.3 \cdot 0.7}$

$$P(2 \leq X \leq 7) = \int_2^7 f(x)\,dx = F(7) - F(2) = pnorm(7, 4.5, \sigma) - pnorm(2, 4.5, \sigma)$$

Exercise 41. Two dice are rolled and we denote $X(Y)$ the sum of points from the two dice (maximum number of points; if the numbers are identical, we take their common value). Compute the correlation of X and Y.

sgr. 2

$$\rho(X,Y)=\frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)\cdot V(Y)}}=\frac{\mathrm{Cov}(X,Y)}{\sigma(X)\cdot\sigma(Y)}$$

$$\Omega=\{(1,1),(1,2),\ldots,(6,6)\}$$

$$|\Omega|=36$$

$$\mathrm{Cov}(X,Y)=E[X\cdot Y]-E[X]\cdot E[Y]$$

Are X and Y independent? $\longrightarrow$ No!

$$P(X=x, Y=y)\neq P(X=x)\cdot P(Y=y) \iff X,Y \text{ - not indep.}$$

$$P(X=x/Y=y)\neq P(X=x) \iff \text{—— " ——}$$

$$X=12 \Longrightarrow Y=6$$

$$X:\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} \end{pmatrix}$$

$3=1+2=2+1$

$4=1+3$
$3+1$
$2+2$

$5=1+4$
$2+3$
$3+2$
$4+1$

$$Y : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{36} & \frac{3}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \end{pmatrix}$$

$(1,2), (2,1), (2,2)$

$(1,3), (2,3), (3,3), (3,2), (3,1)$

$$E[Y] = 1\cdot\frac{1}{36} + 2\cdot\frac{3}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36} = \frac{161}{36} = 4.47$$

$$V[Y] = E[Y^2] - E^2[Y] = \frac{791}{36} - \frac{161^2}{36^2} = \frac{2555}{36^2} = 1.97 \Rightarrow \sigma(Y) = \sqrt{1.97} = 1.40$$

$$E[Y^2] = 1\cdot\frac{1}{36} + 2^2\cdot\frac{3}{36} + 3^2\cdot\frac{5}{36} + \dots + 6^2\cdot\frac{11}{36} = \frac{791}{36}$$

$$E[X] = 7$$

$$V[X] = 5.83 \Rightarrow \sigma(X) = \sqrt{5.83} = 2.41$$

$$X\cdot Y : \left(\begin{array}{cccccccccccccccc} 2\cdot1 & 2\cdot2 & 2\cdot3 & 2\cdot4 & 2\cdot5 & 2\cdot6 & 3\cdot1 & 3\cdot2 & 3\cdot3 & 3\cdot4 & 3\cdot5 & 3\cdot6 & 4\cdot2 & 4\cdot3 & 5\cdot3 & 5\cdot4 \\ \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{36} & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} \end{array}\right.$$

$$P(X\cdot Y = 2\cdot1) = P(X=2, Y=1) = \frac{1}{36}$$

$$\left|\begin{array}{ccccccccccc} 6\cdot3 & 6\cdot4 & 6\cdot5 & 7\cdot4 & 7\cdot5 & 7\cdot6 & 8\cdot4 & 8\cdot5 & 8\cdot6 & 9\cdot5 & 9\cdot6 \\ \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} \end{array}\right.$$

$$\begin{pmatrix} 10\cdot 5 & 10\cdot 6 & 11\cdot 6 & 12\cdot 6 \\ \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} \end{pmatrix}$$

$$X\cdot Y: \begin{pmatrix} 2 & 6 & 8 & 12 & 15 & 20 & 18 & 24 & 30 & 28 & 35 & 42 & 32 & 40 & 48 & 45 & 54 & 50 & 60 & 66 & 72 \\ \frac{1}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} \end{pmatrix}$$

$$X\cdot Y: \begin{pmatrix} 2 & 6 & 8 & 12 & 15 & 18 & 20 & 24 & 28 & 30 & 32 & 35 & 40 & 42 & 45 & 48 & 50 & 54 & 60 & 66 & 72 \\ \frac{1}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} & \frac{2}{36} & \frac{2}{36} & \frac{2}{36} & \frac{1}{36} \end{pmatrix}$$

$$E[X\cdot Y] = 34.22$$

$$Cov(X,Y) = E[X\cdot Y] - E[X]\cdot E[Y] = 34.22 - 7\cdot 4.47 = 2.93$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma(X)\cdot\sigma(Y)} = \frac{2.93}{2.41\cdot 1.40} = \underline{\underline{0.87}} \qquad \rho(X,Y)\in[-1,1]$$

Exercise 85. Calculate the following probabilities both exactly and by using a normal approximation:
(a) $P(X \geq 8)$ where $X \sim Bin(10, 0.7)$
(b) $P(2 \leq X \leq 7)$ where $X \sim Bin(15, 0.3)$
(c) $P(X \leq 4)$ where $X \sim Bin(9, 0.4)$
(d) $P(8 \leq X \leq 11)$ where $X \sim Bin(14, 0.6)$

$\rightarrow$ homework

Exercise 86. Suppose that a fair coin is tossed n times. Estimate the probability that the proportion of heads obtained lies between 0.49 and 0.51 for $n = 100$, 200, 500, 1000 and 2000.

Exercise 81. Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is the probability that the repair time exceeds 4 hours? If it exceeds 4 hours what is the probability that it exceeds 8 hours? What is the expected time required to repair the car?

T - time required to repair a car

$T \sim Exp(\lambda),\ \lambda = \frac{1}{2} \longrightarrow$ PDF: $f(x) = \lambda e^{-\lambda x},\ x \geq 0$

$$P(T>4) = \int_4^\infty f(x)\,dx = \int_4^\infty \frac{1}{2} e^{-\frac{1}{2}x}\,dx = -e^{-1/2 x}\Big|_4^\infty = -\left(0 - e^{-\frac{1}{2}\cdot 4}\right) = e^{-2} = 0.135$$

$$\lim_{x\to\infty} e^{-\frac{1}{2}x} = 0$$

$$P(T>8 \mid T>4) = \frac{P(T>8, T>4)}{P(T>4)} = \frac{P(T>8)}{P(T>4)} = \frac{\int_8^\infty \frac{1}{2}e^{-\frac{1}{2}x}dx}{e^{-2}} = \frac{-e^{-\frac{1}{2}x}\Big|_8^\infty}{e^{-2}} = \frac{e^{-\frac{1}{2}\cdot 8}}{e^{-2}} =$$

$$= \frac{e^{-4}}{e^{-2}} = e^{-2} = 0.135$$

$$E[T] = \int_0^\infty x f(x)\,dx = \int_0^\infty x \cdot \frac{1}{2} e^{-\frac{1}{2}x}\,dx = \int_0^\infty x \cdot \left(-e^{-\frac{1}{2}x}\right)'\,dx =$$

$$= -x e^{-\frac{1}{2}x} \Big|_0^\infty + \int_0^\infty e^{-\frac{1}{2}x}\,dx = -2 e^{-\frac{1}{2}x} \Big|_0^\infty = -2(0-1) = 2$$

Suppose that two chess players had played numerous games and it was determined that the probability that Player A would win is 0.40 , the probability that Player B would win is 0.35 , and the probability that the game would end in a draw is 0.25 . If these two chess players played 12 games, what is the probability that Player A would win 7 games, Player B would win 2 games, and the remaining 3 games would be drawn?" What is the probability that A wins 8 games? What is the expected number of wins for B?

X_1 – nr. of games won by Player A

X_2 – nr. of games won by Player B

X_3 – nr. of games that end in a draw

$X_i \sim Bin(n, p_i)$

$X = (X_1, X_2, X_3) \sim Mult(n, p)$ $n = 12$

$p = (p_1, p_2, p_3)$

$p_1 = 0.40$ (probab. that Player A wins) $n_1 = 7$

$p_2 = 0.35$ (probab. that Player B wins) $n_2 = 2$

$p_3 = 0.25$ (probab. of a draw) $n_3 = 3$

$n_1 + n_2 + n_3 = n$

$P(X = (7,2,3)) = \frac{n!}{n_1! \cdot n_2! \cdot n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3} = \frac{12!}{7! \cdot 2! \cdot 3!} \cdot 0.4^7 \cdot 0.35^2 \cdot 0.25^3 = 0.025$

$$P(\text{"A wins 8 games"}) = P(X_1 = 8) = \binom{n}{8} p_1^8 (1-p_1)^{n-8} = \binom{12}{8} \cdot 0.4^8 \cdot 0.6^4 = 0.042$$

$X_1 \sim Bin(n, p_1)$

$n = 12, \; p_1 = 0.4$

$$\text{Expected } \underbrace{\text{nr. of wins for B}}_{X_2} = n \cdot p_2 = 12 \cdot 0.35 = 4.2 \quad (4 \text{ games})$$

$E[X_2] = n \cdot p_2$

$X_2 \sim Bin(n, p_2)$

(homework)

#: A die is weighted or "loaded", so that the number that appears on each face has pmf:
$P(X=x)=x/21$, for $x=1,2,3,4,5,6$.
If the die is rolled 21 times, compute the probability of rolling one one, two twos, three threes, four fours, five fives and six sixes.

Exercise 83. Suppose you are watching a radioactive source that emits particles at a rate described by an exponential density with parameter $\lambda = 1$. Find the probability that a particle will appear (a) within the next second; (b) within the next 3 seconds; (c) between 3 and 4 seconds from now; (d) after 4 seconds from now. Compute the expected time until the next particle appears.

→ homework

sgr. 5

Exercise 40. We toss two coins, each of them having the number 1 on one face and the number 2 on the other face. We consider the random variable X "the sum of the obtained numbers" and the random variable Y "the maximum of these numbers". Compute the correlation of X and Y.

$Z=(X,Y)$ – random vector

$$\Omega = \{(1,1),(1,2),(2,1),(2,2)\}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}} = \frac{Cov(X,Y)}{\sigma(X)\cdot \sigma(Y)}$$

$$Cov(X,Y) = E[X\cdot Y] - E[X]\cdot E[Y]$$

$$X:\begin{pmatrix} 2 & 3 & 4 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \quad Y:\begin{pmatrix} 1 & 2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$E[X^2] = 4\cdot\frac{1}{4} + 9\cdot\frac{1}{2} + 16\cdot\frac{1}{4} = 1 + \frac{9}{2} + 4 = 5 + \frac{9}{2} = \frac{19}{2}$$

$$E[X] = \cancel{2}\cdot\frac{1}{\cancel{4}_{2}} + 3\cdot\frac{1}{2} + 4\cdot\frac{1}{4} = \frac{1}{2} + \frac{3}{2} + 1 = 2 + 1 = 3$$

$$V[X] = E[X^2] - E^2[X] = \frac{19}{2} - {}^{2/}9 = \frac{1}{2}$$

$$Y:\begin{pmatrix}1 & 2\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix},\quad X:\begin{pmatrix}2 & 3 & 4\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{pmatrix}$$

$$E[Y]=\frac{7}{4}$$

$$V[Y]=E[Y^2]-E^2[Y]=\frac{13}{4}-\frac{49}{16}=\frac{3}{16}$$

$$E[Y^2]=1\cdot\frac{1}{4}+4\cdot\frac{3}{4}=\frac{1}{4}+\frac{12}{4}=\frac{13}{4}$$

$$X\cdot Y:\begin{pmatrix}2\cdot1 & 2\cdot2 & 3\cdot1 & 3\cdot2 & 4\cdot1 & 4\cdot2\\ \frac{1}{4} & 0 & 0 & \frac{2}{4} & 0 & \frac{1}{4}\end{pmatrix},\quad X\cdot Y:\begin{pmatrix}2 & 6 & 8\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{pmatrix}$$

$$P(X=2,Y=1)=\frac{1}{4},\quad P(X=2,Y=2)=0$$

$\underbrace{P(X=2,Y=1)}_{\frac{1}{4}} \neq \underbrace{P(X=2)\cdot P(Y=1)}_{\frac{1}{16}}$ because X and Y are not independent

$$X=2 \Rightarrow Y=1$$

$$E[X\cdot Y]=2\cdot\frac{1}{4}+6\cdot\frac{1}{2}+8\cdot\frac{1}{4}=\frac{1}{2}+\frac{6}{2}+\frac{4}{2}=\frac{11}{2}$$

$$Cov(X,Y)=E[X\cdot Y]-E[X]\cdot E[Y]={}^{2)}\frac{11}{2}-3\cdot\frac{7}{4}=\frac{1}{4}$$

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}}=\frac{\frac{1}{4}}{\sqrt{\frac{1}{2}\cdot\frac{3}{16}}}=\frac{\frac{1}{4}}{\frac{1}{4}\sqrt{\frac{3}{2}}}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$$

$$\rho(X,Y)=\frac{\sqrt{6}}{3}=\underline{0.82}\in[-1,1]$$

Exercise 83. Suppose you are watching a radioactive source that emits particles at a rate described by an exponential density with parameter $\lambda = 1$. Find the probability that a particle will appear (a) within the next second; (b) within the next 3 seconds; (c) between 3 and 4 seconds from now; (d) after 4 seconds from now. Compute the expected time until the next particle appears.

$$\lim_{x\to\infty} e^{-x} = 0$$

T - time of the emition of the next particle.

$T \sim Expo(\lambda)$, $\lambda = 1$, $f(x) = \lambda e^{-\lambda x}$, $x > 0$

(a) $P(T<1) = \int_0^1 f(x)\,dx = \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1 = -(e^{-1}-1) = 1-e^{-1} = 0.632$

(b) $P(T<3) = \int_0^3 f(x)\,dx = -e^{-x}\Big|_0^3 = -(e^{-3}-1) = 1-e^{-3} = 0.95$

(c) $P(T\in(3,4)) = \int_3^4 f(x)\,dx = \int_3^4 e^{-x}\,dx = -e^{-x}\Big|_3^4 = e^{-3}-e^{-4} = 0.031$

(d) $P(T>4) = \int_4^\infty f(x)\,dx = \int_4^\infty e^{-x}\,dx = -e^{-x}\Big|_4^\infty = -(0-e^{-4}) = e^{-4} = 0.018$

$$E[T] = \int_0^\infty x f(x)\,dx = \int_0^\infty x e^{-x}\,dx = \int_0^\infty \underbrace{x}_{f} \cdot \underbrace{(-e^{-x})'}_{g}\,dx =$$

$$= -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = 0 - 0 - e^{-x}\Big|_0^\infty = -(0-1) = 1$$

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = 0$$

Homework:

Exercise 85. Calculate the following probabilities both exactly and by using a normal approximation:
(a) $P(X \geq 8)$ where $X \sim Bin(10, 0.7)$
(b) $P(2 \leq X \leq 7)$ where $X \sim Bin(15, 0.3)$
(c) $P(X \leq 4)$ where $X \sim Bin(9, 0.4)$
(d) $P(8 \leq X \leq 11)$ where $X \sim Bin(14, 0.6)$

Exercise 86. Suppose that a fair coin is tossed n times. Estimate the probability that the proportion of heads obtained lies between 0.49 and 0.51 for $n = 100$, 200, 500, 1000 and 2000.

Exercise 90. The pH levels of food items prepared in a certain way are normally distributed with a standard deviation $\sigma = 0.82$. An experimenter estimates the mean pH level by averaging the pH levels of a random sample of n items. $\rightarrow \bar{X}$
(a) If $n = 5$, what is the probability that the experimenter's estimate is within 0.5 of the true mean value?
(b) If $n = 10$, what is the probability that the experimenter's estimate is within 0.5 of the true mean value?
(c) What sample size n is needed to ensure that there is a probability of at least 99% that the experimenter's estimate is within 0.5 of the true mean value?

$$Bin(n,p) \sim N(\overset{\mu}{np}, \overset{\sigma}{\sqrt{np(1-p)}})$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

85.(a) $P(X \geq 8) = P(X=8) + P(X=9) + P(X=10) = \binom{10}{8} 0.7^8 \cdot 0.3^2 + \binom{10}{9} 0.7^9 \cdot 0.3^1 + \binom{10}{10} 0.7^{10} \cdot 0.3^0$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$= \text{pbinom}(\underbrace{7, 10, 0.7}_{P(X>7)}, \text{lower.tail} = F) = 0.383$

$$P(X \geq 8) \approx \int_8^\infty f(x)\,dx = 1 - F(8) = 1 - \text{pnorm}(8, 10 \cdot 0.7, \sqrt{10 \cdot 0.7 \cdot 0.3}) = 0.245$$

86. $\bar{X}$ – proportion of Heads (sample mean)

$\bar{X} = \frac{X_1+X_2+\dots+X_m}{m}$, X_i – result of the i^{th} toss $\to$ Heads

$X_i \sim Bern(p)$, $p=\frac{1}{2}$, $X_i : \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$P(0.49 < \bar{X} < 0.51) = \int_{0.49}^{0.51} f(x)\,dx = pnorm\left(0.51, \frac{1}{2}, \sqrt{\frac{1}{4m}}\right) - pnorm\left(0.49, \frac{1}{2}, \sqrt{\frac{1}{4m}}\right)$$

Central Limit Theorem $\Rightarrow$ $\bar{X} \sim N(\mu, \sigma)$

$\mu = p$, $\sigma = \sqrt{\frac{p(1-p)}{m}}$

$\mu = E[\bar{X}]$

In a random sample of 20 students at Computer Science, 2nd year, what is the probability 10 students have O blood type, 6 students have A, 2 students have B and the remaining have AB blood type? We know that the probability a person has O is 0.44, A is 0.42, B is 0.10 and AB is 0.04. What is the probability that 4 students have AB blood type? What is the expected number of students that have A blood type?

X_1 - nr. of students with O blood type
X_2 - —"— —"— A —"—
X_3 - —"— —"— B —"—
X_4 - —"— —"— AB —"—

$P(X=(10,6,2,2))=?$
$P(X_4=4)=?$
$X_i \sim Bin(n, p_i)$
$E[X_2]=?$

$X=(X_1,X_2,X_3,X_4) \sim Mult(n,p)$ $n=20$ (sample size)

$p=(p_1,p_2,p_3,p_4)$

$p_1=0.44$ (probab. that a person has O blood type)
$p_2=0.42$ (—"— —"— —"— A —"—)
$p_3=0.10$ (—"— —"— —"— B —"—)
$p_4=0.04$ (—"— —"— AB —"—)

$$P(X=(\underset{m_1}{10},\underset{m_2}{6},\underset{m_3}{2},\underset{m_4}{2})) = \frac{m!}{m_1!\cdot m_2!\cdot m_3!\cdot m_4!}\cdot p_1^{m_1}\, p_2^{m_2}\, p_3^{m_3}\, p_4^{m_4} = \frac{20!}{10!\cdot 6!\cdot 2!^2}\cdot 0.44^{10}\cdot 0.42^{6}\cdot 0.10^{2}\cdot 0.04^{2}$$

$$= \underline{\underline{0.006}}$$

$$P(X=k) = \binom{m}{k} p^k (1-p)^{m-k} = \frac{m!}{k!\cdot(m-k)!}\cdot p^k (1-p)^{m-k}$$

$$P(X_4=4) = \binom{m}{4}\cdot p_4^4\cdot(1-p_4)^{m-4} = \binom{20}{4}\cdot 0.04^4\cdot 0.96^{16} = \underline{\underline{0.006}}$$

$X_4 \sim \mathrm{Bin}(m, p_4)$

$E[X_2] = 0.42\cdot 20 = 4.2\cdot 2 = \underline{8.4}$ (8/9 students)

$X_2 \sim \mathrm{Bin}(m, p_2) \Rightarrow E[X_2] = mp_2$

Exercise 43. We toss a perfectly balanced coin, we denote X_1 the obtained result: $X_1 = 0$ if we obtain the head, $X_1 = 1$ if we obtain the value. If $X_1 = 0$ we toss a tricked coin for which the probability to obtain the value is $2/3$. If $X_1 = 1$ we toss again the balanced coin. We denote X_2 the result of the second throwing ($X_2 = 0$ if we get the head and $X_2 = 1$ if we get the value). Let $X = X_2$ and $Y = X_1 + X_2$.

1. Give the distribution law for X_2
2. Find the joint distribution of the variables X and Y
3. Compute the law of Y conditioned by X.
4. Compute $E(X)$, $E(Y)$, $V(X)$, $V(Y)$, $Cov(X,Y)$, $\rho(X,Y)$.

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$$P(A) = P(A|A_1)P(A_1) + P(A|A_2)P(A_2) + \dots + P(A|A_n)P(A_n)$$

$$P(A_i|A) = \frac{P(A|A_i)P(A_i)}{P(A)}$$

$$X_1 : \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

1. $$X_2 : \begin{pmatrix} 0 & 1 \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$P(X_2 = 0) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12}$$

$$= \underbrace{P(X_2=0|X_1=1)}_{\frac{1}{2}} \cdot \underbrace{P(X_1=1)}_{\frac{1}{2}} + \underbrace{P(X_2=0|X_1=0)}_{\frac{1}{3}} \cdot \underbrace{P(X_1=0)}_{\frac{1}{2}}$$ (law of total probability)

2. $Y: \begin{pmatrix} 0+0 & 0+1 & 1+0 & 1+1 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad Y: \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{7}{12} & \frac{1}{4} \end{pmatrix}$

X_1, X_2 - not indep.

$P(X_1=0, X_2=0) \neq P(X_1=0)\cdot P(X_2=0) \iff X_1, X_2$ - are not indep.

$P(X_1=0, X_2=0) = P(X_2=0 / X_1=0)\cdot P(X_1=0) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

$= P(X_1=0 / X_2=0)\cdot P(X_2=0)$

$P(X_1=0, X_2=1) = P(X_2=1 / X_1=0)\cdot P(X_1=0) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$

$P(X_1=1, X_2=0) = P(X_2=0 / X_1=1)\cdot P(X_1=1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$P(X_1=1, X_2=1) = P(X_2=1 | X_1=1)\cdot P(X_1=1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$$X:\begin{pmatrix}0 & 1\\ \frac{5}{12} & \frac{7}{12}\end{pmatrix},\quad Y:\begin{pmatrix}0 & 1 & 2\\ \frac{1}{6} & \frac{7}{12} & \frac{1}{4}\end{pmatrix}\qquad X=X_2,\ Y=X_1+X_2$$

Joint distribution of X and Y:

$$(X,Y):\begin{pmatrix}(0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2)\\ \frac{1}{6} & \frac{1}{4} & 0 & 0 & \frac{1}{3} & \frac{1}{4}\end{pmatrix}$$

$$P(X=0,Y=0)=\underbrace{P(Y=0\,|\,X=0)\cdot P(X=0)}_{=P(X_2=0,\,X_1=0)}=P(X_1+X_2=0\,|\,X_2=0)\cdot P(X_2=0)=$$

$$=P(X_1=0\,|\,X_2=0)\cdot P(X_2=0)=\frac{P(X_2=0\,|\,X_1=0)\cdot P(X_1=0)}{\cancel{P(X_2=0)}}\cdot\cancel{P(X_2=0)}=P(X_2=0\,|\,X_1=0)\cdot P(X_1=0)=\frac{1}{6}$$

$$=P(X_2=0,\,X_1=1)$$

$$P(X=0,Y=1)=P(X=0\,|\,Y=1)\cdot P(Y=1)=P(X_2=0\,|\,X_1+X_2=1)\cdot P(X_1+X_2=1)=\frac{1}{4}$$

$$=P(X_1+X_2=1\,|\,X_2=0)\cdot P(X_2=0)=P(X_1=1\,|\,X_2=0)\cdot P(X_2=0)=P(X_2=0\,|\,X_1=1)\cdot P(X_1=1)$$

$$P(X=0,Y=2)=P(X_2=0,\,X_1+X_2=2)=0$$

$$P(X=1, Y=1) = P(X_2=1, X_1+X_2=1) = P(X_2=1, X_1=0) = \frac{1}{3}$$

$$P(X=1, Y=2) = P(X_1=1, X_2=1) = \frac{1}{4}$$

$$(X,Y): \begin{pmatrix} (0,0) & (0,1) & (1,1) & (1,2) \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

3. distribution law of Y/X

$X \backslash Y$	0	1	2
0	$\frac{1}{6}$	$\frac{1}{4}$	0
1	0	$\frac{1}{3}$	$\frac{1}{4}$

$$P(Y=0/X=0) = \frac{P(Y=0, X=0)}{P(X=0)} = \frac{\frac{1}{6}}{\frac{5}{12}} = \frac{2}{5}$$

$$P(Y=0/X=1) = \frac{P(Y=0, X=1)}{P(X=1)} = 0$$

$$P(Y=1 \mid X=0) = \frac{P(Y=1, X=0)}{P(X=0)} = \frac{1}{4} \cdot \frac{12}{5} = \frac{3}{5}$$

$$P(Y=1 / X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{\frac{1}{3}}{\frac{7}{12}} = \frac{4}{7}$$

$$P(Y=2 / X=0) = 0$$

$$P(Y=2 / X=1) = \frac{3}{7}$$

4. $X: \begin{pmatrix} 0 & 1 \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}$ $\quad Y: \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{7}{12} & \frac{1}{4} \end{pmatrix}$ $\quad X\cdot Y: \begin{pmatrix} 0 & 0 & 1 & 2 \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$

$X\cdot Y: \begin{pmatrix} 0 & 1 & 2 \\ \frac{5}{12} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$

$E[X] = 0\cdot\frac{5}{12} + 1\cdot\frac{7}{12} = \frac{7}{12}$

$V[X] = E[X^2] - E^2[X] = \frac{7}{12} - \left(\frac{7}{12}\right)^2 = \frac{7}{12}\left(1-\frac{7}{12}\right) = \frac{7}{12}\cdot\frac{5}{12} = \frac{35}{144}$

$E[X^2] = 0^2\cdot\frac{5}{12} + 1^2\cdot\frac{7}{12} = \frac{7}{12}$

$E[Y] = \frac{13}{12}$

$V[Y] = \frac{59}{144}$

$Cov(X,Y) = E[X\cdot Y] - E[X]\cdot E[Y] = \overset{24}{\frac{5}{6}} - \frac{7}{12}\cdot\frac{13}{12} = \frac{120-91}{144} = \frac{29}{144}$

$E[X\cdot Y] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ $\Big|$ $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}} = \frac{\frac{29}{144}}{\sqrt{\frac{35}{144}\cdot\frac{59}{144}}} = \frac{29}{\sqrt{35\cdot 59}} = 0.638$

Suppose a card is drawn randomly from an ordinary deck of playing cards, and then put back in the deck. This exercise is repeated ten times. What is the probability of drawing 4 spade, 1 heart, 3 diamond, and 2 clubs? What is the probability of drawing 7 spades? What is the expected number clubs drawn?

X_1 - nr. of spades drawn
X_2 - nr. of hearts drawn
X_3 - nr. of diamonds drawn
X_4 - nr. of clubs drawn

$X = (X_1, X_2, X_3, X_4) \sim \text{Mult}(n, p)$ $\quad n = 10$

$p = (p_1, p_2, p_3, p_4)$

p_1 - probab. of drawing a spade
p_2 - — " — heart
p_3 - — " — diamond
p_4 - — " — clubs

$p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$

$P(X = (\underset{n_1}{4}, \underset{n_2}{1}, \underset{n_3}{3}, \underset{n_4}{2})) = \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot n_4!} \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$

$n_1 + n_2 + n_3 + n_4 = n$

$= \frac{10!}{4! \cdot 1! \cdot 3! \cdot 2!} \cdot \left(\frac{1}{4}\right)^4 \left(\frac{1}{4}\right)^1 \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^2 = \underline{0.012}$

$$P(X_1=7)=\frac{10!}{7!\cdot 3!}\cdot\left(\frac{1}{4}\right)^7\cdot\left(\frac{3}{4}\right)^3$$

$$X_1 \sim Bin(n, p_1)$$

$$E[X_4]=n\cdot p_4=10\cdot\frac{1}{4}=2.5 \quad (\sim 3 \text{ cards})$$

$$X_4 \sim Bin(n, p_4)$$

$$X_i \sim Bin(n, p_i)$$

Exercise 82. Suppose that the number of years a car will run is exponentially distributed with parameter $\lambda = 1/4$. If you buy a used car today, what is the probability that it will still run after 4 years?

What is number of years the car is expected to run?

X - nr. of years a car will run

$X \sim Expo(\lambda)$, $\lambda = \frac{1}{4}$

$$P(X>4) = \int_4^{\infty} f(x)\,dx = \int_4^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_4^{\infty} = -e^{-\frac{1}{4}\cdot x}\Big|_4^{\infty} = -(0 - e^{-1}) = e^{-1} = 0.368$$

$$f(x) = \lambda e^{-\lambda x},\ x>0$$

$$E[X] = \int_0^{\infty} x f(x)\,dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx = \int_0^{\infty} x \cdot (-e^{-\lambda x})'\,dx = \underbrace{-x e^{-\lambda x}\Big|_0^{\infty}}_{0} + \int_0^{\infty} e^{-\lambda x}\,dx =$$

$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda} = 4$$

Exercise 86. Suppose that a fair coin is tossed n times. Estimate the probability that the proportion of heads obtained lies between 0.49 and 0.51 for $n = 100$, 200, 500, 1000 and 2000.

Exercise 87. Consider a sample of normally distributed random variables $X_1, X_2, \ldots, X_n$ with mean μ and variance $\sigma^2 = 7$. If $n = 15$, what is the probability that $|\mu - \bar{X}| \leq 0.4$? What is the probability if $n = 50$? → Homework

$\bar{X}$ - proportion of heads obtained

$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$, X_i - result of the i^{th} toss

$\bar{X} \sim Bin(n, \frac{1}{2}) \longrightarrow N(\mu, \sigma)$

$p = \frac{1}{2}$ $\qquad$ $\mu = np$, $\sigma = \sqrt{np(1-p)}$

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Exercise 45. A box contains a white balls and b black balls ($a + b \geq 3$). We successively take out balls from the box. Let X, Y and Z be discrete random variables, equal to 1 if the first, the second respectively the third ball extracted is white, and equal to 0 in the contrary. Determine the distribution of the random vector (Y, Z). Find the distribution laws for Y and Z. Compute $Cov(Y, Z)$ and $\rho(Y, Z)$.

$a = 5,\ b = 7$

$$X: \begin{pmatrix} 0 & 1 \\ \frac{7}{12} & \frac{5}{12} \end{pmatrix}, \quad Y: \begin{pmatrix} 0 & 1 \\ \frac{77}{132} & \frac{55}{132} \end{pmatrix}$$

$$P(Y=1) = \overbrace{P(Y=1/X=0)\cdot P(X=0)}^{\frac{7}{12}\cdot\frac{5}{11}} + \overbrace{P(Y=1/X=1)\cdot P(X=1)}^{\frac{5}{12}\cdot\frac{4}{11}} \to \text{law of total probability}$$

- $\frac{7}{12}$ 1st ball is black
  - $\frac{5}{11}$ 2nd ball is white $\to$ w $\to \frac{7}{12}\cdot\frac{5}{11}$
  - $\frac{6}{11}$ 2nd ball is black $\to$ b
- $\frac{5}{12}$ 1st ball is white
  - $\frac{4}{11}$ 2nd b. white $\to$ w $\to \frac{5}{12}\cdot\frac{4}{11}$
  - $\frac{7}{11}$ 2nd b. black $\to$ b

$$P(Y=1) = \frac{7}{12}\cdot\frac{5}{11} + \frac{5}{12}\cdot\frac{4}{11} = \frac{55}{132} ; \quad P(Y=0) = \frac{7}{12}\cdot\frac{6}{11} + \frac{5}{12}\cdot\frac{7}{11} = \frac{77}{132}$$

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$$Z: \begin{pmatrix} 0 & 1 \\ \frac{77}{132} & \frac{55}{132} \end{pmatrix}$$

$$P(Y=0 \mid X=0)\cdot P(X=0) \qquad P(Y=0 \mid X=1)\cdot P(X=1)$$

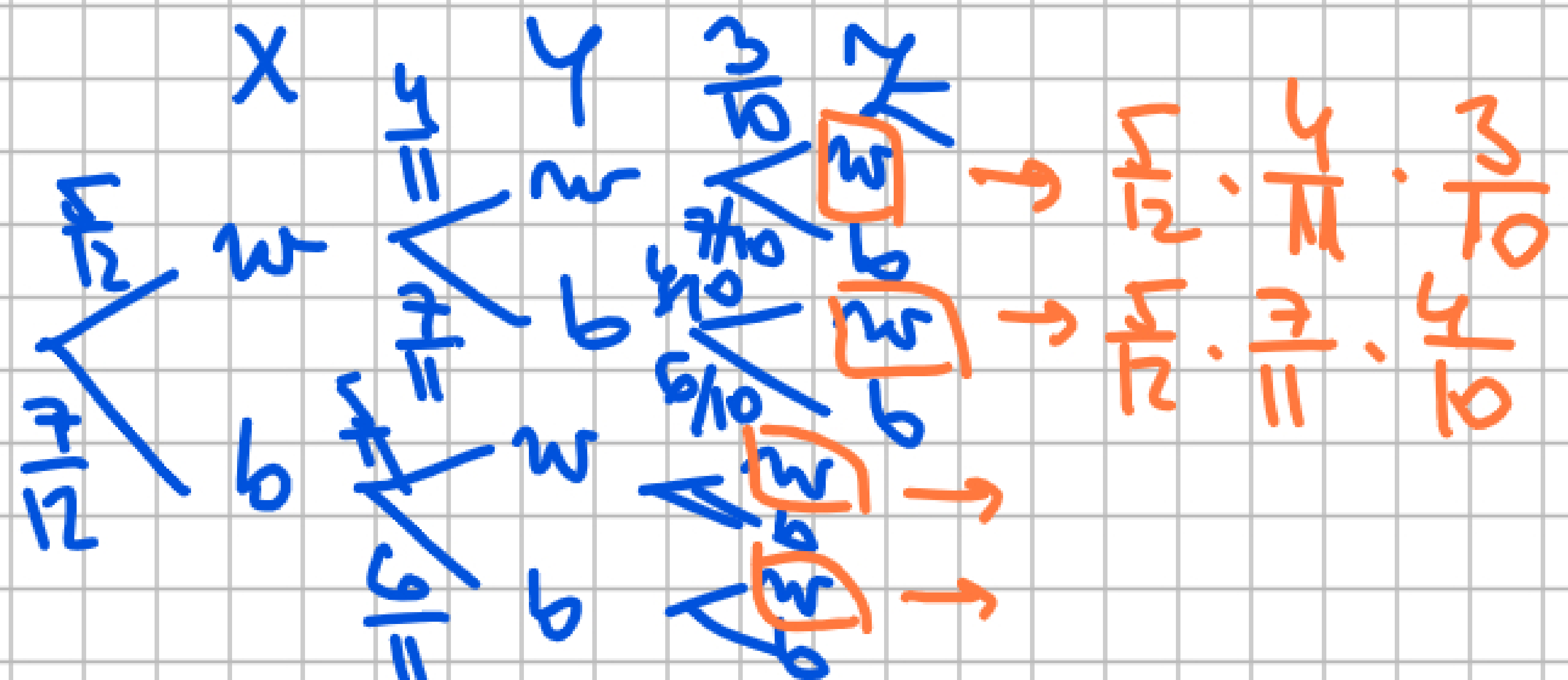
$$P(Z=1) = P(Z=1 \mid X=0, Y=0)\cdot P(X=0,Y=0) + P(Z=1 \mid X=1, Y=0)\cdot P(X=1, Y=0) +$$

$$+ P(Z=1 \mid X=0, Y=1)\cdot P(X=0, Y=1) + P(Z=1 \mid X=1, Y=1)\cdot P(X=1, Y=1) =$$

$$P(Y=1 \mid X=0)\cdot P(X=0) \qquad P(Y=1 \mid X=1)\cdot P(X=1)$$

$$= \frac{5}{10}\cdot\frac{6}{11}\cdot\frac{7}{12} + \frac{4}{10}\cdot\frac{7}{11}\cdot\frac{5}{12} + \frac{4}{10}\cdot\frac{5}{11}\cdot\frac{7}{12} + \frac{3}{10}\cdot\frac{4}{11}\cdot\frac{5}{12} =$$

$$= \frac{210+140+140+60}{1320} = \frac{550}{1320} = \frac{55}{132}$$



$$(Y,Z):\begin{pmatrix}(0,0) & (0,1) & (1,0) & (1,1)\\ \frac{7}{22} & \frac{35}{132} & \frac{35}{132} & \frac{5}{33}\end{pmatrix}$$

$$1-\left(\frac{70}{132}+\frac{7}{22}\right)=1-\frac{112}{132}=\frac{20}{132}=\frac{5}{33}$$

$$P((Y,Z)=(0,0))=P(Y=0,Z=0)=P(Z=0,Y=0/X=0)\cdot P(X=0)+P(Z=0,Y=0/X=1)\cdot P(X=1)$$

Y, Z - are not indep.

$$=\underbrace{P(Z=0/Y=0,X=0)\cdot P(Y=0/X=0)}\cdot P(X=0)+$$

$$+P(Z=0/Y=0,X=1)\cdot P(Y=0/X=1)\cdot P(X=1)=$$

$$=\frac{5}{10}\cdot\frac{6}{11}\cdot\frac{7}{12}+\frac{6}{10}\cdot\frac{7}{11}\cdot\frac{5}{12}=\frac{210+210}{1320}=\frac{420}{1320}=\frac{42}{132}=\frac{7}{22}$$

$$P(Y=0,Z=1)=P(Z=1/Y=0,X=0)\cdot P(Y=0/X=0)\cdot P(X=0)+P(Z=1/Y=0,X=1)\cdot P(Y=0/X=1)\cdot P(X=1)$$

$$=\frac{5}{10}\cdot\frac{6}{11}\cdot\frac{7}{12}+\frac{4}{10}\cdot\frac{7}{11}\cdot\frac{5}{12}=\frac{210+140}{1320}=\frac{350}{1320}=\frac{35}{132}$$

$$P(Y=1,Z=0)=P(Z=0/Y=1,X=0)\cdot P(Y=1/X=0)\cdot P(X=0)+P(Z=0/Y=1,X=1)\cdot P(Y=1/X=1)\cdot P(X=1)=$$

$$=\frac{6}{10}\cdot\frac{5}{11}\cdot\frac{7}{12}+\frac{7}{10}\cdot\frac{4}{11}\cdot\frac{5}{12}=\frac{35}{132}$$

$$(Y,Z): \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ \frac{7}{22} & \frac{35}{132} & \frac{35}{132} & \frac{5}{33} \end{pmatrix} \qquad Y: \begin{pmatrix} 0 & 1 \\ \frac{77}{132} & \frac{55}{132} \end{pmatrix} \qquad Z: \begin{pmatrix} 0 & 1 \\ \frac{77}{132} & \frac{77}{132} \end{pmatrix}$$

$$\operatorname{Cov}(Y,Z) = E[Y\cdot Z] - E[Y]\cdot E[Z] = \frac{5}{33} - \left(\frac{55}{132}\right)^2 = -\frac{35}{1584}$$

$$E[Y] = E[Z] = 0\cdot\frac{77}{132} + 1\cdot\frac{55}{132} = \frac{55}{132} = \frac{5}{12}$$

$$Y\cdot Z: \begin{pmatrix} 0\cdot 0 & 0\cdot 1 & 1\cdot 0 & 1\cdot 1 \\ \frac{7}{22} & \frac{35}{132} & \frac{35}{132} & \frac{5}{33} \end{pmatrix}, \quad Y\cdot Z: \begin{pmatrix} 0 & 1 \\ \frac{28}{33} & \frac{5}{33} \end{pmatrix}$$

$$P(Y=0, Z=0) = \frac{7}{22}$$

$$\frac{7}{22} + \frac{70}{132} = \frac{\overset{6}{7}}{22} + \frac{\overset{2}{35}}{66} = \frac{112}{132} = \frac{28}{33}$$

$$E[Y\cdot Z] = \frac{5}{33}$$

$$\rho(Y,Z)=\frac{\operatorname{Cov}(Y,Z)}{\sqrt{V(Y)\cdot V(Z)}}$$

$$Y:\begin{pmatrix} 0 & 1 \\ \frac{77}{132} & \frac{55}{132} \end{pmatrix}$$

$$\quad \frac{7}{12}'' \qquad ''\frac{5}{12}$$

$$V(Y)=E[Y^2]-E^2[Y]=\frac{55}{132}-\left(\frac{55}{132}\right)^2=\frac{55}{132}\left(1-\frac{55}{132}\right)=\frac{55}{132}\cdot\frac{77}{132}=\frac{5}{12}\cdot\frac{7}{12}=\frac{35}{144}$$

$$E[Y^2]=0^2\cdot\frac{77}{132}+1^2\cdot\frac{55}{132}=\frac{55}{132}=\frac{5}{12}$$

$$V(Z)=\frac{35}{144}$$

$$\rho(Y,Z)=\frac{-\frac{35}{1584}}{\sqrt{\frac{35}{144}\cdot\frac{35}{144}}}=-\frac{\frac{\cancel{35}}{1584}}{\frac{\cancel{35}}{144}}=-\frac{144}{1584}=-\frac{1}{11}=-0.09$$

Suppose we have a bowl with 10 marbles - 2 red marbles, 3 green marbles, and 5 blue marbles. We randomly select 9 marbles from the bowl, with replacement. What is the probability of selecting 2 green marbles and 4 blue marbles? What is the probability of selecting 3 red marbles? What is the expected number of blue marbles selected?

X_1 - nr. of red marbles

X_2 - nr. of green marbles

X_3 - nr. of blue marbles

$X = (X_1, X_2, X_3) \sim Mult(n, p) \implies X_i \sim Bin(n, p_i), \; i = \overline{1,3}$

$n = 9, \; p = (p_1, p_2, p_3)$

$p_1 = \frac{1}{5}$ - probab. of selecting a red marble

$p_2 = \frac{3}{10}$ - " - green - " -

$p_3 = \frac{1}{2}$ - " - blue - " -

$$P(X = (3,2,4)) = \frac{n!}{n_1! \cdot n_2! \cdot n_3!} \cdot p_1^{n_1} \, p_2^{n_2} \, p_3^{n_3} =$$

$n_1 = 4$

$n_2 = 2$

$n_3 = 3$

$n_1 + n_2 + n_3 = n$

$$= \frac{9!}{3! \cdot 2! \cdot 4!} \cdot \left(\frac{1}{5}\right)^3 \cdot \left(\frac{3}{10}\right)^2 \cdot \left(\frac{1}{2}\right)^4 =$$

$$= \ldots$$

$$P(X_1 = 3) = \binom{n}{3} p_1^3 (1-p_1)^{n-3} = \binom{9}{3} \cdot \left(\frac{1}{5}\right)^3 \cdot \left(\frac{4}{5}\right)^6 = \dots$$

$$E[X_3] = n \cdot p_3 = 9 \cdot \frac{1}{2} = 4.5 \quad (\sim 5 \text{ blue marbles})$$

Exercise 86. Suppose that a fair coin is tossed n times. Estimate the probability that the proportion of heads obtained lies between 0.49 and 0.51 for $n = 100$, 200, 500, 1000 and 2000.

Exercise 87. Consider a sample of normally distributed random variables $X_1, X_2, \ldots, X_n$ with mean μ and variance $\sigma^2 = 7$. If $n = 15$, what is the probability that $|\mu - \bar{X}| \leq 0.4$? What is the probability if $n = 50$?

→ Homework

Exercise 80. The lifetime of an electronic tube is a continuous random variable X with exponential density $f(x) = b\,e^{-bx}$ $x \geq 0$. Find $P(k \leq X \leq k+1)$ and the expected lifetime of the tube. → H.